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Breakdown of Acceleration Waves in Radiative Magneto-fluids

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ABSTRACT

The problem of propagation of acceleration waves in an optically thick medium of electrically conducting fluid has been dealt with. During propagation of the waves, the effects of radiation pressure, radiation energy density, and heat transfer through thermal radiation and thermal conduction have been taken into account. The growth equation for the variation of amplitude of the wave has been derived and solved. It has been concluded that all the compressive waves with initial amplitudes greater than a critical value will grow and terminate into a shock wave due to nonlinear steepening, while all expansion waves will decay out. Acritical stage, when the compressive wave will either grow or decay, has also been discussed. The effects of radiation pressure and radiative heat transfer on the shock formation have been discussed and analysed.

Keywords: Wave propagation, acceleration waves, shock waves, shock wave formation, radiative heat transfer, nonlinear wave, thermal conduction, non-equilibrium flows

1. INTRODUCTION

In the recent advances of space technology as hypersonic flights, powerplants for space exploration, gas-cooled nuclear reactors (where the temperature is very high and the density is low), the study of propagation and formation of shock waves in fluids becomes an interesting problem. One of the most interesting part of this study is that it is a subclass of nonlinear waves which admit analytical solutions. When a disturbance is produced in hot plasma, the thermal conduction effect plays an essential role in the determination of the behaviour of finite-amplitude wave heads. So in a gaseous flow under high temperature conditions, it is more realistic to take into account the thermal conduction effects. Becker[1], Bowen and Chen[2] studied the various properties of the acceleration waves in non-equilibrium flows. Several researchers[3–7] have studied the various properties of the weak nonlinear waves and have provided answers to the questions when a weak wave breaks down and a shock wave is formed. However, less emphasis is put on the thermal conduction effect, and therefore, the exact behaviour of the waves in radiation gas dynamics is not completely understood.

The problem investigates the essential features of the effects of radiation energy density, radiation pressure, and radiative heat flux on the global behaviour of the acceleration waves propagating in electrically conducting fluids permeated by a magnetic field. The analysis is based upon the theory of singular surfaces, which, in comparison to the theory of characteristics, quickly leads to results of general significance. An optically thick gas medium is considered with such high temperature and low pressure that the radiation pressure number

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R„ is not negligible, but the profiles structured by radiant heat transfer can be assumed to be imbedded in the discontinuties. Pai[8] suggested that under the approximation of local thermodynamic equilibrium, the radiative heat flux term is similar to that of heat conduction. In this case, the effective thermal conductivity is given by

$$K_{eff} = K + 4D_R a_R T^3$$

where K, D_R, a,, and T are the coefficients of thermal conduction, Rosseland diffusion constant, Stefan-Boltzmann constant, and the absolute gas temperature, respectively.

Let a discontinuity surface $\Sigma(t)$ be considered across which the flow and field variables are continuous but their first and higher-order derivatives are discontinuous. Such a discontinuity is defined as a weak discontinuity or an acceleration wave. The boundary conditions are:

$$[p]=0,[\rho]=0,[u]=0,[T]=0,[h]=0$$

$$\left[\frac{\partial p}{\partial r}\right]\neq 0,\left[\frac{\partial \rho}{\partial r}\right]\neq 0,\left[\frac{\partial u}{\partial r}\right]\neq 0,\left[\frac{\partial h}{\partial r}\right]\neq 0 \qquad (1$$

where, [Z] denotes the discontinuity in the quantity enclosed. The geometrical and the kinematical conditions[9] for a singular surface $\Sigma(t)$ are of the order that can be expressed in the form:

$$\left[\frac{\partial z}{\partial r}\right]=B,\left[\frac{\partial z}{\partial t}\right]=-BG \qquad 2(a)$$

$$\left[\frac{\partial^2 z}{\partial r^2}\right]=\bar{B},\left[\frac{\partial^2 z}{\partial t^2}\right]=-\bar{B}G+\frac{\delta B}{\delta t} \qquad 2(b)$$

where, [Z] refers for any of the flow variables, B is a scalar function defined over $\Sigma(t)$, G is the velocity of the surface $\Sigma(t)$ into a uniform medium at rest, and $\delta/\delta t$ is the operator of time derivative as observed from the wavefront itself.

2. PROPAGATION LAW

The basic equations governing an axisymmetric magnetohydrodynamics flow under an optically thick gas approximation[8] with thermal conduction and radiation effects are:

$$\frac{\partial p}{\partial t}+u\frac{\partial p}{\partial r}+\rho\frac{\partial u}{\partial r}+\frac{apu}{r}=0 \qquad (3)$$

$$\rho\frac{\partial u}{\partial t}+\rho u\frac{\partial u}{\partial r}+\frac{\partial}{\partial r}(h+p_R)+\alpha(1-n)\frac{2h}{r}=0 \qquad (4)$$

$$\frac{\partial h}{\partial t}+u\frac{\partial h}{\partial r}+2h\frac{\partial u}{\partial r}+\frac{2\alpha nhu}{r}=0 \qquad (5)$$

$$\rho\frac{\partial}{\partial t}\left(\frac{\gamma}{\gamma-1}\frac{p}{\rho}+4\frac{p_R}{\rho}\right)+\rho u\frac{\partial}{\partial r}\left(\frac{\gamma}{\gamma-1}\frac{p}{\rho}+4\frac{p_R}{\rho}\right)$$
$$-\frac{\partial}{\partial t}(p+p_R)-u\frac{\partial}{\partial r}(p+p_R)+\frac{\partial}{\partial r}\left(K_{eff}\frac{\partial T}{\partial r}\right)=0 \qquad (6)$$

Using the relations

$$p^R=\frac{1}{3}E_R=\frac{1}{3}a_R T^4=p.R_p=C_v(\gamma-1)\rho T$$

In Eqns *(6)* and (4), one gets:

$$\rho\frac{\partial u}{\partial t}+\rho u\frac{\partial u}{\partial r}+\frac{\partial h}{\partial r}+(1+4R_p)\frac{\partial p}{\partial r}$$
$$-4R_p\frac{p}{\rho}\frac{\partial \rho}{\partial r}+\frac{\alpha(1-n)2h}{r}=0 \qquad (7)$$

and

$$\{1+12(\gamma-1)R_p\}\frac{\partial p}{\partial t}-\{\gamma+16(\gamma-1)R_p\}\frac{p}{\rho}.\frac{\partial \rho}{\partial t}$$
$$+\{1+12(\gamma-1)\}u\frac{\partial p}{\partial r}-\{\gamma+16(\gamma-1)R_p\}u\frac{p}{\rho}.\frac{\partial \rho}{\partial r}$$
$$+(\gamma-1)\frac{\partial}{\partial r}\left(K_{eff}\frac{\partial T}{\partial r}\right)=0 \qquad (8)$$

The caloric equation of state yields:

$$\frac{1}{p}\frac{\partial p}{\partial r}=\frac{1}{\rho}\frac{\partial \rho}{\partial r}+\frac{1}{T}\frac{\partial T}{\partial r} \tag{9}$$

Here, ρ, p, u, h, R_p, γ, and C_v denote the density, the gas pressure, the gas velocity, the magnetic pressure, the radiation pressure number, (R_p= radiation **pressure/gas** pressure) the heat exponent of the gas, and the specific heat at constant volume, respectively. a is a parameter of geometrical symmetry and takes the values 0, I, and 2, for planar, cylindrical, and spherical symmetry, respectively. The parameter n takes the values 0 and I for axial and azimuthal magnetic field, and r denotes the distance from the centre of symmetry.

Taking jumps in Eqns (3), (7) and (5) and making the use of Eqn (1) and (2(a)), one gets:

$$(u-G)\zeta+\rho\lambda=0 \tag{10}$$

$$\rho(u-G)\lambda+\eta+(1+4R_p)\xi-4\frac{p}{\rho}R_p\zeta=0 \tag{11}$$

$$(u-G)\eta+2h\lambda=0 \tag{12}$$

where

$$\lambda=\left[\frac{\partial u}{\partial r}\right],\zeta=\left[\frac{\partial \rho}{\partial r}\right],\xi=\left[\frac{\partial p}{\partial r}\right],\eta=\left[\frac{\partial h}{\partial r}\right]$$

From the law of conservation of energy across a discontinuity surface[10] $\Sigma(t)$, one has:

$$\left[\frac{\partial T}{\partial r}\right]=0 \tag{13}$$

Applying the compatibility condition [Eqn 2(a) in Eqn (9)], one gets the relation:

$$\xi=a_0^2\zeta, \tag{14}$$

where a, is the isothermal speed of sound and the suffix 0 denotes the value just ahead of the wavefront.

From Eqns (10)–(12), one gets:

$$\lambda=\frac{-(u-G)\zeta}{\rho}=\frac{-(u-G)\eta}{2h}=\frac{-(1+4R_p)(u-G)\xi}{\left\{\rho(u-G)^2-2h+4pR_p\right\}} \tag{15}$$

Using Eqn (15) in Eqn (11), one gets:

$$\lambda\left\{\rho(u-G)^2-2h-(1+4R_p)\rho a_0^2+4pR_p\right\}=0 \tag{16}$$

The assumption that $\Sigma(t)$ is a discontinuity surface of order, one implies that λ = 0. Hence, one gets:

$$(u-G)^2=a_0^2+b_0^2 \tag{17}$$

It has been assumed that the medium ahead of the propagating surface $\Sigma(t)$ is uniform and at rest. For this case, the speed of propagation is given by

$$G^2=a_0^2+b_0^2 \tag{18}$$

where b_0 is the Alfven speed of sound = $2h/\rho$.

In view of the conditions of constant and the rest states ahead of the wave, the relation [Eqn (15)] reduces to the form:

$$\lambda=\frac{G\zeta}{\rho}=\frac{G\eta}{2h}=\frac{G(1+4R_p)\xi}{\left\{\rho G^2-2h+4pR_p\right\}} \tag{19}$$

3. GROWTH EQUATIONS

Differentiating Eqns (3), (7) and (5) wrt r and applying the compatibility conditions [Eqn 2(b)] for the jump, one gets:

$$\frac{\delta\zeta}{\delta t}=G\bar{\zeta}-2\zeta\lambda-\rho\bar{\lambda}-\frac{\alpha\rho\lambda}{R} \tag{20}$$

$$\rho\frac{\delta\lambda}{\delta t}=\rho G\bar{\lambda}-(1+4R_p)\bar{\xi}-\bar{\eta}+\frac{4}{\rho}R_p\xi^2+4\frac{p}{\rho}R_p\bar{\zeta}-4\frac{p}{\rho^2}R_p\zeta^2\frac{-2\alpha(1-n)\eta}{R} \tag{21}$$

$$\frac{\delta\eta}{\delta t} = G\bar{\eta} - 3\eta\lambda - 2h\bar{\lambda} - \frac{2\alpha nh\lambda}{R} \tag{22}$$

where

$$\bar{\lambda} = \left[\frac{\partial^2 u}{\partial r^2}\right], \bar{\zeta} = \left[\frac{\partial^2 \rho}{\partial r^2}\right], \bar{\xi} = \left[\frac{\partial^2 p}{\partial r^2}\right], \bar{\eta} = \left[\frac{\partial^2 h}{\partial r^2}\right]$$

Taking the jump of the Eqn (8), one has:

$$-\{1 + 12(\gamma-1)R_p\}\xi G + \{\gamma + 16(\gamma-1)R_p\}\zeta\frac{p}{\rho}G$$
$$+(\gamma-1)K_{eff}\bar{\theta} = 0 \tag{23}$$

where

$$\bar{\theta} = \left[\frac{\partial^2 T}{\partial r^2}\right]$$

Differentiating Eqn (9) wrt r and taking jump, one gets:

$$\bar{\theta} = \frac{\bar{\xi} - (\gamma-1)C_v\bar{\zeta}T}{(\gamma-1)\rho C_v} \tag{24}$$

Using Eqns (20), (22) and (24) in the Eqn (21), one gets:

$$\frac{\delta\lambda}{\delta t} = -G\lambda\left\{\frac{\mu_1}{R} + \mu_2\right\} - \frac{\lambda^2}{G}\mu_3 \tag{25}$$

where

$$\mu_1 = \frac{1}{G^2}\{\alpha(a_0^2 + nb_0^2) - 2\alpha(1-n)b_0^2\}$$

$$\mu_2 = \frac{(\gamma-1)^2\rho C_v(1+4R_p)^2}{K_{eff}G}$$

$$\mu_3 = \frac{1}{G}(a_0^2 + \frac{3}{2}b_0^2)$$

The Eqn (25) is the fundamental differential equation for the variation of λ along the normal trajectories of the wavefront $\Sigma(t)$. This differential equation governs the growth and decay of the acceleration wave.

4. GLOBAL BEHAVIOUR OF AMPLITUDE

Now the amplitude $\lambda(t)$ of an acceleration wave is defined by

$$\lambda(t) = \left[\frac{\partial u}{\partial r}\right]$$

If $R_0 = \Sigma(t_0)$ represents the initial position of the wave at t = 0, the position of the wave at time t is given by

$$R = R_0 + Gt \tag{26}$$

In view of the Eqn (26), one has:

$$\frac{\delta\lambda}{\delta t} = G\frac{d\lambda}{d\sigma} \tag{27}$$

where $\sigma = R - R_,$.

Using Eqns (26) and (27) in the Eqn (25), one gets:

$$\frac{d\lambda}{d\sigma} + \lambda\left\{\frac{\mu_1}{R} + \mu_2\right\} + \lambda^2\frac{\mu_3}{G^2} = 0 \tag{28}$$

Since the medium ahead of the wave is uniform and at rest, one has:

$$R = \frac{1 + k(0)\sigma}{k(0)} \tag{29}$$

where $k(0)$ is the initial curvature of the wave.

Using Eqn (29) in the Eqn (28), the solution of the growth equation [Eqn (28)] can be written as

$$\lambda(\sigma) = \frac{\exp(-\mu_2\sigma)(1 + k(0)\sigma)^{-\mu_1}}{\frac{1}{\lambda(0)} + \frac{\mu_3}{G^2}\int_0^\sigma \exp(-\mu_2 x)(1 + k(0)x)^{-\mu_1}dx} \tag{30}$$

where $\lambda(0)$ is the initial wave amplitude at time t_0. The solution [Eqn (30)] shows that the amplitude $\lambda(\sigma)$ of an expansion wave $[\lambda(0)>0]$ decreases in time and tends to become zero as $\sigma \to \infty$. On the other hand, the amplitude of a compressive wave $[\lambda(0)<0]$ will, in general, grow and tend to become infinity after a finite time.

In this case, an acceleration wave will breakdown and a shock-type discontinuity will be formed after a finite critical time t_c, provided the initial wave amplitude $\lambda(0)$ is numerically greater than the critical λ_c given by

$$\lambda_c = \left\{ \frac{\mu_3}{G^2} \int_0^\infty \exp(-\mu_2\sigma)(1+k(0)\sigma)^{-\mu_1} d\sigma \right\}^{-1} \quad (31)$$

If $|\lambda(0)| < \lambda_c$, then even a compressive wave will decay and no shock-type discontinuity will be formed. The critical time t_c for the shock formation can be determined by the relation:

$$\int_0^{\sqrt{a_0^2+b_0^2}\,t_c} \exp(-\mu_2\sigma)(1+k(0)\sigma)^{-\mu_1} d\sigma = \frac{a_0^2+b_0^2}{\mu_3|\lambda(0)|} \quad (32)$$

Differentiating Eqns (31) and (32) partially wrt the radiation pressure number R_p and keeping other parameter constant, one gets:

$$\frac{\partial\lambda_c}{\partial R_p} = \left\{ \frac{8\mu_3(1+4R_p)(\gamma-1)^2\rho C_v}{G^3\lambda_c^2 K_{eff}} \right\} \times \int_0^\infty xe^{-\mu_2 x}(1+k(0))^{-\mu_1} dx \geq 0 \quad (33)$$

$$\frac{\partial t_c}{\partial R_p} = \frac{8\rho C_v(\gamma-1)^2(1+4R_p)}{G^2 K_{eff}} e^{\mu_2 G t_c}(1+k(0)Gt_c)^{\mu_1} \times \int_0^{Gt_c} xe^{-\mu_2 x}(1+k(0)x)^{-\mu_1} dx \geq 0 \quad (34)$$

It is clear from Eqns (33) and (34) that the radiation pressure number effect increases the critical value of initial amplitude above which there occurs a shock formation and below which there is no shock formation. It also increases the critical time t_c, and thus, has a stabilising effect, and delays the process of shock formation. Similarly, if one partially differentiates Eqns (31) and (32) wrt K_{eff}, one gets:

$$\frac{\partial\lambda_c}{\partial K_{eff}} = \frac{-\mu_3\rho C_v(\gamma-1)^2(1+4R_p)^2}{G^3 K_{eff}^2\lambda_c^2} \times \int_0^\infty xe^{-\mu_2 x}(1+k(0)x)^{-\mu_1} dx \leq 0. \quad (35)$$

$$\frac{\partial t_c}{\partial K_{eff}} = \frac{-\rho C_v(\gamma-1)^2}{G^2 K_{eff}^2} e^{\mu_2 G t_c}(1+k(0)Gt_c)^{\mu_1} \times \int_0^{Gt_c} xe^{-\mu_2 x}(1+k(0)x)^{-\mu_1} dx \leq 0. \quad (36)$$

It is clear from Eqns (35) and (36) that the effects of thermal and radiative heat transfer decrease the critical amplitude λ_c and the critical time t, and thus have destabilising effect in the sense that these accelerate the process of breakdown of weak waves and formation of shock-type discontinuities.

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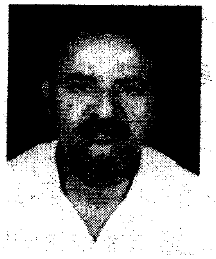

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